ANY FLAT BUNDLE ON A PUNCTURED DISC HAS AN OPER STRUCTURE

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Abstract. We prove that any flat G-bundle, where G is a complex connected reductive algebraic group, on the punctured disc admits the structure of an oper. This result is important in the local geometric Langlands correspondence proposed in [FG]. Our proof uses certain deformations of the affine Springer fibers which could be of independent interest. As a byproduct, we construct representations of affine Weyl groups on the homology of these deformations generalizing representations constructed by Lusztig.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$, $\mathfrak{g} = \mathrm{Lie}(G)$. Let $F = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. In this note we prove that every flat G-bundle on the formal punctured disc $D^\times = \mathrm{Spec}F$ has an oper structure. This proves Conjecture 10.1.1 of [Fr1] (see also [Fr2], Conjecture 1).

By definition, a *flat G-bundle* (equivalently, de Rham G-local system) on $D^\times$ is a principal G-bundle on $D^\times$ with a connection, which is automatically flat. In concrete terms, the set of isomorphism classes of flat G-bundles is the quotient

$$\mathrm{Loc}_G(D^\times) = \mathfrak{g}(F)/G(F), \tag{1}$$

where $G(F)$ acts on its Lie algebra $\mathfrak{g}(F)$ by gauge transformations as follows:

$$\mathrm{Ga}_g(A) = \mathrm{Ad}_g(A) - (\partial_t g)g^{-1}, \quad \text{for } A \in \mathfrak{g}(F), g \in G(F). \tag{2}$$

The meaning of the expression

$$d\log(g) := (\partial_t g)g^{-1} \tag{3}$$

as an element in $\mathfrak{g}(F)$ is spelled out, e.g., in [Fr1] §1.2.4.

Let $B \subset G$ be a Borel subgroup. We recall [BD] that a *G-oper* is a flat G-bundle with a reduction to B satisfying certain conditions. Let us describe the set of isomorphism classes of G-opers on $D^\times$ in concrete terms. Choose a maximal torus $T \subset B$ and let $\mathfrak{t} \subset \mathfrak{b}$ be the corresponding inclusion of Lie algebras. Let I_{f} be the set of vertices in the finite Dynkin diagram corresponding to G. Let $\alpha_i \in \mathfrak{t}^*, i \in I_{\mathrm{f}}$ be the set of simple roots and $X_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ be a non-zero root vector corresponding to $-\alpha_i$. (Here, for a

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root $\beta \in \mathfrak{t}^*$, we write $\mathfrak{g}_\beta$ for the corresponding root subspace of $\mathfrak{g}$.) Then the space of G-opers on $D^\times$ is the quotient

$$\mathrm{Op}_G(D^\times) = \left\{ \sum_{i \in I_{\mathrm{f}}} \psi_i X_{-\alpha_i} + v \;\middle|\; \psi_i \in F^\times, v \in \mathfrak{b}(F) \right\} \Big/ B(F), \tag{4}$$

where the action of $B(F)$ is given by (2). Note that if G is semisimple of adjoint type, then $T(F)$ acts simply transitively on the space of the $\psi_i, i \in I_{\mathrm{f}}$. Hence the quotient (4) is isomorphic to

$$\mathrm{Op}_G(D^\times) = \left\{ \sum_{i \in I_{\mathrm{f}}} X_{-\alpha_i} + v \;\middle|\; v \in \mathfrak{b}(F) \right\} \Big/ N(F), \tag{5}$$

where $N = [B, B]$ is the unipotent radical of B.

There is an obvious forgetful map

$$\mathrm{Op}_G(D^\times) \to \mathrm{Loc}_G(D^\times), \tag{6}$$

taking the $B(F)$-gauge equivalent classes to $G(F)$-gauge equivalent classes.

The main result of this note is

Theorem 1. *The map* (6) *is surjective.*

This statement is important in the local geometric Langlands correspondence developed by D. Gaitsgory and the first author [FG] (see [Fr1] for an exposition). According to [FG], to each flat ${}^LG$-bundle σ on $D^\times$ one should be able to assign a category $\mathcal{C}_\sigma$ equipped with an action of the formal loop group $G(F)$ (here ${}^LG$ is the Langlands dual group of G, which in this paragraph is assumed to be a simply-connected semisimple complex algebraic group, so that ${}^LG$ is of adjoint type). These categories should satisfy some universality property. In [FG] a candidate for $\mathcal{C}_\sigma$ was proposed. Namely, let χ be a pre-image of σ in $\mathrm{Op}_{{}^LG}(D^\times)$ under the map (6), with G replaced by ${}^LG$ (provided that it exists). Then $\mathcal{C}_\sigma$ should be equivalent to the category of modules over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ of critical level with central character determined by χ. This category is equipped with a natural action of $G(F)$. However, for this prescription to work for all σ it is necessary for the map (6) to be surjective.

Remark 1. A flat GL_n-bundle on $D^\times$ is the same as a rank n vector bundle $\mathcal{F}$ on $D^\times$ with a connection ∇. $(\mathcal{F}, \nabla)$ has an oper structure if and only if there exists $\phi \in \Gamma(D^\times, \mathcal{F})$ such that $\phi, \nabla\phi, \ldots, \nabla^{n-1}\phi$ generate $\mathcal{F}$. Such ϕ is called a cyclic vector of $(\mathcal{F}, \nabla)$. Therefore, the statement of Theorem 1 for $G = GL_n$ means that any flat rank n vector bundle on $D^\times$ has a cyclic vector. This statement is proved in [D], pp. 42–43.

Remark 2. Let us recall Kostant's theorem [Ko]. Set $f = \sum_{i \in I_{\mathrm{f}}} X_{-\alpha_i}$. Kostant proved that every *regular* orbit of $\mathfrak{g}$ intersects with $f + \mathfrak{b}$. In other words, the map

$$\{f + v \mid v \in \mathfrak{b}\}/N \to \mathfrak{g}^{\mathrm{reg}}/G$$

is surjective (in fact, an isomorphism), where $\mathfrak{g}^{\mathrm{reg}}/G$ denotes the GIT quotient. Therefore, Theorem 1 may be viewed as an analogue of Kostant's theorem for connections

on the punctured disc (compare with formula (5)). An important difference is that a connection can be brought into an oper form *without any regularity assumption.*

Remark 3. The statement analogous to Theorem 1 for a smooth projective curve X of genus greater than zero is known to be false. For instance, if G is of adjoint type, there is a unique (up to an isomorphism) G-bundle on X that can carry an oper structure (see [BD] §3.5). However, it is expected that any flat G-bundle on X has an oper structure with regular singularities at finitely many points.

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2. Proof of the main theorem in the case when A_r is regular nilpotent

We begin our proof of Theorem 1. Let $A \in \mathfrak{g}(F)$. By taking r small enough, we can always assume that A may be written as

$$A = A_r t^r + A_{r+1} t^{r+1} + \cdots, \qquad r < -1, \quad A_r \text{ is nilpotent.}$$

Here A_r can be zero. First, we have

Lemma 2. *If A_r is regular nilpotent, then there exists some $g \in G(\mathcal{O})$ such that $\mathrm{Ga}_g(A)$ is an oper.*

Proof. Without loss of generality, we can, and will, assume that $A_r = f = \sum_{i \in I_{\mathrm{f}}} X_{-\alpha_i}$. Let $e \in \mathfrak{b}$ be the unique element such that $\{e, 2\check{\rho}, f\}$ is a principal $\mathfrak{sl}_2$-triple. Let $\mathfrak{g}^e$ be the centralizer of e in $\mathfrak{g}$. Let $G^{(1)}(\mathcal{O})$ be the first congruence subgroup of $G(\mathcal{O})$, i.e. the kernel of the evaluation map $G(\mathcal{O}) \to G$. We prove that there is some $g \in G^{(1)}(\mathcal{O})$ such that $\mathrm{Ga}_g(A) \in ft^r + \mathfrak{g}^e(F)$, which is in the oper form.

According to representation theory of $\mathfrak{sl}_2$, we have $\mathfrak{g} = \mathfrak{g}^e + \mathrm{ad} f(\mathfrak{g})$. Therefore, there exists $X_1 \in \mathfrak{g}$ such that $A_{r+1} + [X_1, f] \in \mathfrak{g}^e$. Let $g_1 = \exp(tX_1)$. Since $r < -1$,

$$\mathrm{Ga}_{g_1}(A) = ft^r + (A_{r+1} + [X_1, f])t^{r+1} + \widetilde{A}_{r+2} t^{r+2} + \cdots.$$

Next, there exists some $X_2 \in \mathfrak{g}$ such that $\widetilde{A}_{r+2} + [X_2, f] \in \mathfrak{g}^e$. Let $g_2 = \exp(t^2 X_2)$. Again, since $r < -1$,

$$\mathrm{Ga}_{g_2}(\mathrm{Ga}_{g_1}(A)) = ft^r + (A_{r+1} + [X_1, f])t^{r+1} + (\widetilde{A}_{r+2} + [X_2, f])t^{r+2} + \widetilde{A}_{r+3} t^{r+3} + \cdots$$

By induction, we can find $g_1, \ldots, g_{k-1}$ such that the coefficients of $t^{r+1}, \ldots, t^{r+k-1}$ of $\mathrm{Ga}_{g_{k-1}} \cdots \mathrm{Ga}_{g_1}(A)$ are in $\mathfrak{g}^e$. Let $\widetilde{A}_{r+k}$ be the coefficient of t^{r+k} in $\mathrm{Ga}_{g_{k-1}} \cdots \mathrm{Ga}_{g_1}(A)$. Let $X_k \in \mathfrak{g}$ such that $\widetilde{A}_{r+k} + [X_k, f] \in \mathfrak{g}^e$ and let $g_k = \exp(t^k X_k)$. Then the coefficient of t^{r+k} in $\mathrm{Ga}_{g_k} \cdots \mathrm{Ga}_{g_1}(A)$ belongs to $\mathfrak{g}^e$, while the coefficients of $t^r, \ldots, t^{r+k-1}$ remain unchanged. Let $g = \cdots g_k \cdots g_2 g_1$. This is a well-defined element in $G^{(1)}(\mathcal{O})$ which satisfies the requirement of the lemma. □

Remark 4. Let A_r be an arbitrary regular element of $\mathfrak{g}$. By Kostant's theorem (see Remark 2), we can assume, without loss of generality, that $A_r = f + v, v \in \mathfrak{b}$. By a slight modification of the above argument, we can then also prove that there exists

$g \in G^{(1)}(\mathcal{O})$ such that $\mathrm{Ga}_g(A)$ is an oper. Thus, we obtain a simple proof of the statement of Theorem 1 in the case when the leading term A_r is regular. The real challenge is to prove that it holds even without this assumption. $\square$

By the previous lemma, in order to prove Theorem 1 it suffices to prove that there exists $g \in G(F)$ such that $B = \mathrm{Ga}_g(A) = B_r t^r + B_{r+1}t^{r+1} + \cdots$, with B_r regular nilpotent. Recall that we are under the assumption $r < -1$. The rest of this paper is devoted to proving this fact.

3. Deformed affine Springer fibers

If $A = A_r t^r + A_{r+1}t^{r+1} + \cdots$ with $A_r \neq 0$, we call r the *order* of A, and sometimes denote it by $\mathrm{ord}(A)$. Let

$$M_A = \{g \in G(F) \mid \mathrm{ord}(\mathrm{Ga}_{g^{-1}}(A)) \geq \mathrm{ord}(A) = r\}.$$

This is a subset of elements g of $G(F)$ which is the set of solutions of certain algebraic equations on the coefficients of g. Hence it is clear that it is the set of points of an ind-subscheme of $G(F)$. It is clearly invariant under the right multiplication by elements of the subgroup $G(\mathcal{O})$. Therefore the quotient

$$Y_A := M_A/G(\mathcal{O})$$

is a well-defined closed ind-subscheme of the affine Grassmannian $\mathrm{Gr} = G(F)/G(\mathcal{O})$. We call it the *deformed affine Springer fiber* associated to A.

Let us explain this terminology. Set $\widetilde{A} = t^{-r}A \in \mathfrak{g}(\mathcal{O})$. For $\lambda \in \mathbb{C}$, let

$$Y_{\widetilde{A},\lambda} = \{g \in G(F), \mathrm{Ad}_{g^{-1}}(\widetilde{A}) - \lambda t^{-r} d\log(g^{-1}) \in \mathfrak{g}(\mathcal{O})\}/G(\mathcal{O}),$$

where $d\log(g)$ is defined as in (3). Then $Y_{\widetilde{A},1} = Y_A$, and $Y_{\widetilde{A},0}$ is the affine Springer fiber of $\widetilde{A}$ defined by Kazhdan and Lusztig in [KL2] (see also [GKM1]). Let us first show that

Lemma 3. *For any $gG(\mathcal{O}) \in Y_A$,*

$$(\mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r} d\log(g^{-1}) \mod t) \in \mathfrak{g}(\mathcal{O})/t\mathfrak{g}(\mathcal{O}) = \mathfrak{g}$$

is nilpotent.

Proof. Let T be the maximal torus of G whose Lie algebra is $\mathfrak{t}$. Let $X_*(T)$ be the coweight lattice of T and $X_*(T)_+$ be semi-group of dominant coweights. Each $\check{\lambda} \in X_*(T)$ defines a point $t^{\check{\lambda}} \in T(F) \subset G(F)$. We have the Birkhoff decomposition

$$G(F) = \bigsqcup_{\check{\lambda} \in X_*(T)_+} G(\mathcal{O}) t^{\check{\lambda}} G(\mathcal{O}).$$

Let $g \in G(F)$ be as in the lemma. We can write it as $g = g_1 t^{\check{\lambda}} g_2$ for $g_1, g_2 \in G(\mathcal{O})$ and a dominant coweight $\check{\lambda}$. Then we have

$$\mathrm{Ga}_{g_2^{-1}}(B) = \mathrm{Ga}_{t^{\check{\lambda}}} \mathrm{Ga}_{g_1}(A).$$

It is clear that

$$C = \mathrm{Ga}_{g_1}(A) = C_r t^r + C_{r+1}t^{r+1} + \cdots ,$$
$$D = \mathrm{Ga}_{g_2^{-1}}(B) = D_r t^r + D_{r+1}t^{r+1} + \cdots ,$$

with C_r nilpotent. We need to show that D_r is nilpotent.

Let $\mathfrak{g} = \sum_i \mathfrak{g}_i$ be the weight decomposition of $\mathfrak{g}$ with respect to $\check{\lambda}$. Then $\mathfrak{g}_{\geq 0} := \sum_{i\geq 0} \mathfrak{g}_i$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{g}_{>0} := \sum_{i>0} \mathfrak{g}_i$ is its nil-radical and $\mathfrak{g}_0$ is a Levi subalgebra. Similarly, one has $\mathfrak{g}_{\leq 0}$ and $\mathfrak{g}_{<0}$. We observe that $X = X_0 + X_{>0} \in \mathfrak{g}_0 + \mathfrak{g}_{>0}$ (resp. $X = X_0 + X_{<0} \in \mathfrak{g}_0 + \mathfrak{g}_{<0}$) is nilpotent in $\mathfrak{g}$ if and only if X_0 is nilpotent in $\mathfrak{g}_0$.

Now since $D = \mathrm{Ga}_{t^{\check{\lambda}}}(C)$, we know that $C_r \in \mathfrak{g}_{\geq 0}$ and $D_r \in \mathfrak{g}_{\leq 0}$. Furthermore, if we decompose $C_r = C' + C''$ with $C' \in \mathfrak{g}_0, C'' \in \mathfrak{g}_{>0}$ and $D = D' + D''$ with $D' \in \mathfrak{g}_0, D'' \in \mathfrak{g}_{<0}$, then $C' = D'$. Since C_r is nilpotent, $C' = D'$ is nilpotent. Therefore D_r is nilpotent. □

By Lemma 2, Theorem 1 holds if there is a point $gG(\mathcal{O}) \in Y_A$ such that the above element is regular nilpotent. Such a point $gG(\mathcal{O})$ is called a *regular point* of Y_A (cf. [GKM2]). Therefore, the main theorem follows from

Theorem 4. *If A_r is nilpotent (equivalently, $\widetilde{A} \mod t$ is nilpotent) and $r \leq -2$, then Y_A has a regular point.*

Let us interpret this theorem more geometrically. Let I be the Iwahori subgroup of $G(F)$, i.e. the pre-image of $B \subset G$ under the evaluation map $G(\mathcal{O}) \to G$, and $\mathcal{F}\ell = G(F)/I$ be the affine flag variety. Without loss of generality, we can assume that $\widetilde{A} \mod t \in \mathfrak{n}$, where $\mathfrak{n}$ is the nil-radical of $\mathfrak{b}$. Let

$$X_A = \{g \in G(F) \mid \mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r} d\log(g^{-1}) \in \mathrm{Lie} I\}/I \in \mathcal{F}\ell.$$

There is a natural projection $\pi : X_A \to Y_A$. The following lemma is clear.

Lemma 5. *A point $p = gG(\mathcal{O}) \in Y_A$ is a regular point if and only if $\pi^{-1}(p)$ consists of a single point.*

Observe that to prove Theorem 4, it is enough to prove that $Y_A \cap \mathrm{Gr}^0$ has a regular point, where Gr^0 is the neutral component of Gr. Let $\tilde{G}$ be the simply-connected cover of the derived group of G, and write $A = A_0 + A_1$, where $A_0 \in \mathrm{Lie}(Z(G)^0)(F)$ ($Z(G)^0$ being the neutral component of the center $Z(G)$ of G) and $A_1 \in \mathrm{Lie}(\tilde{G})(F)$. Then $Y_A \cap \mathrm{Gr}^0$ is (topologically) isomorphic to Y_{A_1}, and $X_A \cap \mathcal{F}\ell^0$ is (topologically) isomorphic to X_{A_1}. In addition, the projection $\mathcal{F}\ell^0 \to \mathrm{Gr}^0$ is (topologically) isomorphic to the map $\mathcal{F}\ell_{\tilde{G}} \to \mathrm{Gr}_{\tilde{G}}$, where $\mathcal{F}\ell_{\tilde{G}}$ (resp. $\mathrm{Gr}_{\tilde{G}}$) denotes the affine flag variety (resp. affine Grassmannian) of $\tilde{G}$. Therefore, the map $X_A \cap \mathcal{F}\ell^0 \to Y_A \cap \mathrm{Gr}^0$ is (topologically) isomorphic to $X_{A_1} \to Y_{A_1}$. Then according to Lemma 5, it is sufficient to prove Theorem 4 for connected simply-connected semisimple algebraic groups. Hence, from now on, we will assume that G is a connected simply-connected semisimple algebraic group.

An analogous statement for non-deformed affine Springer fibers has been proved in [KL2] §4. By imitating their proof, we find that it is sufficient to prove two propositions. The first one is the following:

Proposition 6. *Y_A is finite-dimensional.*

Next, we formulate the second proposition. Recall that the affine Weyl group W_{aff} of $G(F)$ acts on $H_*(\mathcal{F}\ell)$ (cf. [Ka] §2.7), where $H_*(\cdot)$ stands for the Borel–Moore homology.

Proposition 7. *Assume that A_r is nilpotent (possibly, equal to zero) and $r \leq -2$. Then the image of $H_*(X_A) \to H_*(\mathcal{F}\ell)$ is invariant under the action of the affine Weyl group W_{aff}.*

For the sake of completeness, let us repeat the argument from [KL2] §4 that shows how the above two propositions imply Theorem 4.

Let $d = \dim X_A$ (it is finite by Proposition 6). Let X be an irreducible component of X_A of dimension d. Denote by $[X] \in H_{2d}(\mathcal{F}\ell)$ the homology class represented by X. Then $[X] \neq 0$ by *loc. cit.* §4, Lemma 6. Let V_A be the image of $H_{2d}(X_A) \to H_{2d}(\mathcal{F}\ell)$. Then V_A is generated by these $[X]$. By Proposition 7, V_A is a subrepresentation of the representation of W_{aff} on $H_{2d}(\mathcal{F}\ell)$. By *loc. cit.* §4, Lemma 8, V_A has a non-zero invariant vector under the action of the finite Weyl group $W_{\mathrm{f}} \subset W_{\mathrm{aff}}$. For $i \in I_{\mathrm{f}}$, let P_i be the parahoric subgroup of $G(F)$ with Lie algebra $\mathrm{Lie}P_i = \mathrm{Lie}I + \mathfrak{g}_{-\alpha_i}$. Let $\mathcal{F}\ell_i$ be the partial affine flag variety of parahoric subgroups of $G(F)$ which are conjugate to P_i, and $\pi_i : \mathcal{F}\ell \to \mathcal{F}\ell_i$ be the projection. Assume that Y_A does not contain a regular point. Then for any $p = gI \in X_A$, $(\mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r} d\log(g^{-1}) \mod t)$ is an element of $\mathfrak{n}$ (by Lemma 3) which is not regular, and therefore is contained in the nil-radical of some parabolic subalgebra $\mathfrak{p}_i = \mathfrak{b} + \mathfrak{g}_{-\alpha_i}$, $i \in I_{\mathrm{f}}$. In this case, for any $g' \in gP_i$, $(\mathrm{Ad}_{g'^{-1}}(\widetilde{A}) - t^{-r} d\log(g'^{-1}) \mod t)$ is also contained in $\mathfrak{n}$ (in fact, in the nil-radical of $\mathfrak{p}_i$) and therefore $\pi_i^{-1}(\pi_i(p)) \subset X_A$. For each d-dimensional irreducible component $X \subset X_A$, let $X_i, i \in I_{\mathrm{f}}$ be the closed subset of points p on X such that $\pi_i^{-1}(\pi_i(p)) \subset X_A$. Then $X = \cup_{i \in I_{\mathrm{f}}} X_i$. Since X is irreducible, $X = X_i$ for some i, i.e., there exists some $i \in I_{\mathrm{f}}$ such that $X = \pi_i^{-1}(\pi_i(X))$. Let T_{s_i} be the corresponding simple reflection in W_{f}, which acts on $H_{2d}(\mathcal{F}\ell)$. Then $(\mathrm{Id} + T_{s_i})[X] = 0$. Now let $T = \sum_{w \in W_{\mathrm{f}}} T_w$. Since for any $i \in I_{\mathrm{f}}$, $T = Q_i(\mathrm{Id} + T_{s_i})$, we find that $T[X] = 0$ for any d-dimensional irreducible component $X \subset X_A$. Therefore, $TV_A = 0$, which contradicts the fact that V_A has a non-zero invariant vector under the action of W_{f}.

In the remaining part of this note we prove Propositions 6 and 7 about the deformed affine Springer fibers. We also discuss the action of the affine Weyl group on $H_*(X_A)$.

4. Proof of Proposition 6

We begin with the proof of Proposition 6. Recall the definition of M_A from the beginning of last section. For any $g \in M_A$, we consider the following $\mathbb{C}$-vector space

$$T_g = \{X \in \mathfrak{g}(F) | \partial_t X + [B, X] \in t^r \mathfrak{g}(\mathcal{O})\} / \mathfrak{g}(\mathcal{O})$$

where $B = \mathrm{Ga}_{g^{-1}}(A)$. Observe that if $g' = gg_1$ with $g_1 \in G(\mathcal{O})$, then $B' = \mathrm{Ga}_{g'^{-1}}(A) = \mathrm{Ga}_{g_1^{-1}}(B)$, and there is a canonical isomorphism $\gamma_{g_1} : T_g \cong T_{g'}$ given by $X \mapsto \mathrm{Ad}_{g_1^{-1}}(X)$.

Therefore, T_g is canonically attached to every $gG(\mathcal{O}) \in Y_A$. From the definition of Y_A, it is clear that T_g is canonically isomorphic to the tangent space of Y_A at $gG(\mathcal{O})$. We claim that the dimension of this $\mathbb{C}$-vector space is $\leq (-r)\dim\mathfrak{g}$. This proves that the dimension of Y_A is $\leq (-r)\dim\mathfrak{g}$.

We regard $\mathfrak{g}(F)$ as a vector space over F, with a connection $\nabla_t = \partial_t + \mathrm{ad}(B)$. Then $T_g = \nabla^{-1}(t^r\mathfrak{g}(\mathcal{O}))/\mathfrak{g}(\mathcal{O})$. Now the claim is a direct consequence of the following lemma, whose proof was suggested to us by D. Arinkin.

Let (V, ∇) be a finite-dimensional vector space over F with a connection. By an $\mathcal{O}$-*lattice* in V we understand a finite generated $\mathcal{O}$-submodule L of V such that the natural map $L \otimes_{\mathcal{O}} F \to V$ is an isomorphism. By a *lattice* in V we understand a $\mathbb{C}$-subspace in V that is commensurable with an $\mathcal{O}$-lattice.

Lemma 8. *For any lattice $L \subset V$, $\nabla^{-1}(L)$ is also a lattice of V, and the relative dimension of $\nabla^{-1}(L)$ to L is*

$$[\nabla^{-1}(L) : L] := \dim\frac{\nabla^{-1}(L)}{\nabla^{-1}(L)\cap L} - \dim\frac{L}{\nabla^{-1}(L)\cap L} \leq 0.$$

Remark 5. This lemma is an easy consequence of Deligne's theory of "good lattices" for connections (cf. [D] pp.110-112), as we learned from D. Arinkin. However, to prove the existence of "good lattices", Deligne used the existence of the cyclic vector for (V, ∇) (cf. Remark 1). Therefore we prefer to avoid using these results in the proof of our theorem.

Proof. We first recall that the connection (V, ∇) is said to be in the *canonical form* (with respect to some F-basis $\mathbf{e}$ of V), if it looks as follows:

$$\partial_t + H_1 t^{r_1} + H_2 t^{r_2} + \cdots + H_m t^{r_m} + X t^{-1},$$

where $r_1 < r_2 < \cdots < r_m < -1$, H_i are diagonal matrices, X is an upper triangular matrix, and $[H_i, X] = 0$. It is proved in [BV], §6 that, possibly after a finite field extension E/F, for every connection (V, ∇), there exists some (E-)basis $\mathbf{e}$ of $V \otimes_F E$, such that this connection is in a canonical form with respect to this basis.

Now we begin to prove the lemma. Assume that $\dim V = n$. Let $E = F(t^{1/d})$ be a finite extension of F and an E-basis $\mathbf{e}$ of $V \otimes_F E$ such that the connection ∇ with respect to this basis is in the canonical form. Let $\Lambda = \mathcal{O}_E\mathbf{e}$, where $\mathcal{O}_E$ is the integral closure of $\mathcal{O}$ in E. Λ is a lattice of $V \otimes_F E$. Since the connection is in the canonical form with respect to $\mathbf{e}$, we have:

$$\nabla^{-1}(t^k\Lambda) \subset t^{k+1}\Lambda + \mathrm{Sol}, \quad \text{for any } k, \tag{7}$$

where $\mathrm{Sol} \subset V \otimes_F E$ is the kernel of ∇ (that is, the solution space of ∇). Note that $\dim_{\mathbb{C}}(\mathrm{Sol}) \leq n$.

Set $M = \Lambda \cap V$ inside $V \otimes_F E$. Then M is a lattice in V. By (7),

$$[\nabla^{-1}(t^k M) : t^k M] \leq 0,$$

because the codimension of $t^{k+1}M$ in $V \cap (t^{k+1}\Lambda + \mathrm{Sol})$ is at most n.

Finally, we can prove the statement. Indeed, take any lattice L and choose k such that $L \supset t^k M$. Since

$$\dim(L/t^k M) \geq \dim(\nabla^{-1}L/\nabla^{-1}(t^k M)),$$

the statement follows.

□

Remark 6. Observe that the tangent spaces for the non-deformed affine Springer fiber are never finite-dimensional (even for regular semisimple elements in $\mathfrak{g}(F)$). This is because the non-deformed affine Springer fiber of a regular semisimple element is highly non-reduced and has infinitely many "nilpotent directions".

5. Proof of Proposition 7

Let $I_{\mathrm{aff}} = I_{\mathrm{f}} \bigsqcup \{i_0\}$ be the set of vertices in the affine Dynkin diagram for $G(F)$, with i_0 corresponding to the affine vertex. Denote by T_{s_i} to the simple reflection corresponding to the vertex $i \in I_{\mathrm{aff}}$. It is enough to construct, for each i, an involution $\sigma_i : H_*(X_A) \to H_*(X_A)$ such that the natural map $j : H_*(X_A) \to H_*(\mathcal{F}\ell)$ satisfies $j(\sigma_i(x)) = T_{s_i} j(x)$.

Let $\mathrm{Aut}^0(D)$ be the group of automorphisms of $D = \mathrm{Spec}\mathcal{O}$. It is an extension of $\mathbb{G}_m$ by a pro-unipotent group $\mathrm{Aut}^+(D)$ (see, e.g., [FB] §6.2). The Lie algebra $\mathrm{Der}^0(D)$ of $\mathrm{Aut}^0(D)$ is topologically spanned by $\{t^n\partial_t; n \geq 1\}$ and the Lie algebra $\mathrm{Der}^+(D)$ of $\mathrm{Aut}^+(D)$ is topologically spanned by $\{t^n\partial_t; n \geq 2\}$. $\mathrm{Aut}^0(D)$ acts on $G(F)$, and we can form the semi-direct product $G(F) \rtimes \mathrm{Aut}^0(D)$. We have

$$\mathrm{Lie}(G(F) \rtimes \mathrm{Aut}^0(D)) = \mathfrak{g}(F) \oplus \mathrm{Der}^0(D) \quad \text{as vector spaces.}$$

Obviously, the action of $\mathrm{Aut}^0(D)$ on $G(F)$ leaves $G(\mathcal{O})$ invariant. Therefore, it acts on Gr. We thus obtain an action of $G(F) \rtimes \mathrm{Aut}^0(D)$ on Gr. In a similar fashion, $G(F) \rtimes \mathrm{Aut}^0(D)$ acts on all the affine (partial) flag varieties of $G(F)$, as is seen from the following lemma.

A standard parahoric subgroup of $G(F)$ is a parahoric subgroup of $G(F)$ that contains I. For $i \in I_{\mathrm{aff}}$, let P_i be the standard minimal parahoric subgroup corresponding to i.

Lemma 9. *The action of* $\mathrm{Aut}^0(D)$ *on* $G(F)$ *leaves* $I, P_i, i \in I_{\mathrm{aff}}$ *invariant and therefore leaves all standard parahoric subgroups of* $G(F)$ *invariant.*

Proof. Write $G(\mathcal{O}) = G^{(1)}(\mathcal{O})G$. The action of $\mathrm{Aut}^0(D)$ on $G(\mathcal{O})$ leaves $G^{(1)}(\mathcal{O})$ invariant and fixes G. Since, I and $P_i, i \in I_{\mathrm{f}}$ are pre-images of subgroups of G under the evaluation map $G(\mathcal{O}) \to G$, they are invariant under the action of $\mathrm{Aut}^0(D)$. It remains to show that P_{i_0} is also invariant under the action of $\mathrm{Aut}^0(D)$, where i_0 is the affine vertex in the affine Dynkin diagram of $\mathfrak{g}$.

We have $\mathrm{Lie}P_{i_0} = \mathrm{Lie}I + t^{-1}\mathfrak{g}_\theta$, where $\mathfrak{g}_\theta$ the the root space corresponding to the highest root θ. It is clear that $[\mathrm{Der}^0(D), t^{-1}\mathfrak{g}_\theta] \subset t^{-1}\mathbb{C}[[t]]\mathfrak{g}_\theta \subset \mathrm{Lie}P_{i_0}$. Therefore, the action of $\mathrm{Aut}^0(D)$ also leaves P_{i_0} invariant. Since the standard parahoric subgroups are generated by some of the P_i's, the lemma follows. □

Thus, elements in the Lie algebra $\mathrm{Lie}(G(F) \rtimes \mathrm{Aut}^0(D))$ act on these affine (partial) flag varieties by vector fields. The zero sets of these vector fields are nothing but our deformed affine Springer fibers! The reason is the following. The group $G(F)$ acts on $\mathrm{Lie}(G(F) \rtimes \mathrm{Aut}^0(D))$ via the adjoint representation. Let us denote this adjoint

representation by $\widetilde{\mathrm{Ad}}$ to distinguish it from the adjoint representation of $G(F)$ on $\mathfrak{g}(F)$. Let

$$(\widetilde{A}, t^{-r}\partial_t) \in \mathfrak{g}(F) \oplus \mathrm{Der}^0(D), \qquad r \leq -1.$$

We have

Lemma 10. *For $g \in G(F)$, $\widetilde{\mathrm{Ad}}_g((\widetilde{A}, t^{-r}\partial_t)) = (\mathrm{Ad}_g(\widetilde{A}) - t^{-r}(\partial_t g)g^{-1}, t^{-r}\partial_t)$.*

Proof. Let $B \in \mathfrak{g}(F)$. We have

$$\begin{aligned}
[\widetilde{\mathrm{Ad}}_g((\widetilde{A}, t^{-r}\partial_t)), B] &= \widetilde{\mathrm{Ad}}_g[(\widetilde{A}, t^{-r}\partial_t), \mathrm{Ad}_{g^{-1}}(B)] \\
&= [\mathrm{Ad}_g(\widetilde{A}), B] + \mathrm{Ad}_g[t^{-r}\partial_t, \mathrm{Ad}_{g^{-1}}(B)] \\
&= [\mathrm{Ad}_g(\widetilde{A}), B] + \mathrm{Ad}_g([t^{-r}\partial_t(g^{-1})g, \mathrm{Ad}_{g^{-1}}(B)] + \mathrm{Ad}_{g^{-1}}(t^{-r}\partial_t B)) \\
&= [(\mathrm{Ad}_g(\widetilde{A}) - t^{-r}(\partial_t g)g^{-1}, t^{-r}\partial_t), B].
\end{aligned}$$

Since $\mathfrak{g}$ is semisimple, this identity implies the desired formula. $\square$

Therefore, if $\widetilde{A}, r$ are as in the assumption of Theorem 4, we obtain that the reduced algebraic variety $X_A^{\mathrm{red}} \subset \mathcal{F}\ell$ underlying X_A is the zero set of the vector field on $\mathcal{F}\ell$ obtained by the action of $(\widetilde{A}, t^{-r}\partial_t) \in \mathrm{Lie}(G(F) \rtimes \mathrm{Aut}^0(D))$. Likewise, $Y_A^{\mathrm{red}} \subset \mathrm{Gr}$ is the zero set of the corresponding vector field on Gr. Let $\mathcal{F}\ell_i = G(F)/P_i$, and $\pi_i : \mathcal{F}\ell \to \mathcal{F}\ell_i$ be the projection. This is a $\mathbb{P}^1$-fibration. We will also define $X_{A,i}^{\mathrm{red}}$ to be the zero set of the corresponding vector field on $\mathcal{F}\ell_i$. It is clear that the projection $\pi_i : \mathcal{F}\ell \to \mathcal{F}\ell_i$ restricts to $\pi_i : X_A^{\mathrm{red}} \to X_{A,i}^{\mathrm{red}}$.

Now, under the assumptions of Theorem 4, $r \leq -2$, and $\widetilde{A} \in \mathrm{Lie}I^0$, where $I^0 = [I, I]$ is the pro-unipotent radical of I. Therefore,

$$(\widetilde{A}, t^{-r}\partial_t) \in \mathrm{Lie}I^0 \oplus \mathrm{Der}^+(D) = \mathrm{Lie}(I^0 \rtimes \mathrm{Aut}^+(D)).$$

Since $I^0 \rtimes \mathrm{Aut}^+(D)$ is pro-unipotent, the vector field on $\mathcal{F}\ell$ (resp., on $\mathcal{F}\ell_i$) gives rise to an action of $\mathbb{G}_a$ on $\mathcal{F}\ell$ (resp., on $\mathcal{F}\ell_i$). Furthermore, the projection $\pi_i : \mathcal{F}\ell \to \mathcal{F}\ell_i$ is $\mathbb{G}_a$-equivariant. Now $X_{A,i}^{\mathrm{red}}$ is just the fixed point set of this $\mathbb{G}_a$ action on $\mathcal{F}\ell_i$. Therefore, there is a fiberwise $\mathbb{G}_a$-action on $\pi_i^{-1}(X_{A,i}^{\mathrm{red}})$, which is a $\mathbb{P}^1$-fibration over $X_{A,i}^{\mathrm{red}}$, and X_A^{red} is just the fixed point set. Now the construction of [KL1] §2 gives us the desired involution $\sigma_i : H_*(X_A) \to H_*(X_A)$.

This completes the proof of Proposition 7 and hence of Theorem 4. Therefore Theorem 1 is now proved.

6. The action of the affine Weyl group on $H_*(X_A)$

We continue to assume that G is a connected simply-connected semisimple complex algebraic group. Let A be a regular semisimple nil-element in $\mathfrak{g}(F)$, i.e., $(\mathrm{ad}\, A)^r \to 0$ if $r \to \infty$, as defined in [KL2] §2. According to *loc. cit.*, this is equivalent to the property that A is conjugate to an element of $\mathfrak{g}(\mathcal{O})$ whose reduction modulo t is a nilpotent element of $\mathfrak{g}$. Let Sp_A be the *non-deformed* affine Springer fiber of A in $\mathcal{F}\ell$. In [Lu] §5, Lusztig constructed an action of W_{aff} on $H_*(\mathrm{Sp}_A)$. We show in this section that a similar construction can be applied to obtain an action of W_{aff} on the homology of the *deformed* affine Springer fibers.

Let $(\widetilde{A}, t^{-r}\partial_t) \in \mathrm{Lie}I^0 \oplus \mathrm{Der}^+(D)$. We will prove that the homology $H_*(X_A)$ itself admits an action of the affine Weyl group, where the simple reflection corresponding to i will act on $H_*(X_A)$ by σ_i constructed above. The only new result here is Proposition 12, the counterpart of which for the usual affine Springer fiber is proved in [Lu] §5.4.

For every $J \subsetneqq I_{\mathrm{aff}}$, let P_J be the standard parahoric subgroup of $G(F)$, generated by P_i, $i \in J$. This is a pro-algebraic group. Let P_J^u be its pro-unipotent radical, so that $G_J := P_J/P_J^u$ is a reductive group. Let $\mathfrak{g}_J = \mathrm{Lie}G_J$. For example, if $J = I_{\mathrm{f}}$ is the set of vertices in the finite Dynkin diagram, then $P_{I_{\mathrm{f}}} = G(\mathcal{O})$, $P_{I_{\mathrm{f}}}^u = G^{(1)}(\mathcal{O})$ and $G_{I_{\mathrm{f}}} = G$. The construction is based on the following

Lemma 11. *Let $J \subsetneqq I_{\mathrm{aff}}$. Then for any $g \in P_J$ and $r \leq -2$ we have $t^{-r}(\partial_t g)g^{-1} \in \mathrm{Lie}P_J^u$.*

Proof. It is enough to show that in $\mathrm{Lie}(G(F) \rtimes \mathrm{Aut}^0(D))$, $[t^{-r}\partial_t, \mathrm{Lie}P_J] \subset \mathrm{Lie}P_J^u$. It is easy to see that

$$t^2\mathfrak{g}(\mathcal{O}) \subset \mathrm{Lie}P_J^u \subset \mathrm{Lie}P_J \subset t^{-1}\mathfrak{g}(\mathcal{O})$$

for any $J \subsetneqq I_{\mathrm{aff}}$. First we assume that $i_0 \notin J$. In this case, $\mathrm{Lie}P_J \subset \mathfrak{g}(\mathcal{O})$ and therefore $[t^{-r}\partial_t, \mathrm{Lie}P_J] \subset t^2\mathfrak{g}(\mathcal{O}) \subset \mathrm{Lie}P_J^u$. The lemma holds. Next, we assume that $J = \{i_0\} \cup J'$, with $J' \subsetneqq I_{\mathrm{f}}$. Then

$$\mathrm{Lie}P_J = \mathrm{Lie}P_J \cap \mathfrak{g}(\mathcal{O}) + \sum_{\beta \in \Delta^+_{J'} \cup \{0\}} t^{-1}\mathfrak{g}_{\theta-\beta},$$

where $\Delta^+_{J'}$ is the set of positive roots for $G_{J'}$. Clearly, $[t^{-r}\partial_t, \mathrm{Lie}P_J \cap \mathfrak{g}(\mathcal{O})] \in \mathrm{Lie}P_J^u$. In addition, $[t^{-r}\partial_t, t^{-1}\mathfrak{g}_{\theta-\beta}] = t^{-r-2}\mathfrak{g}_{\theta-\beta}$, which belongs to $\mathrm{Lie}P_J$. But since $t^{r+2}\mathfrak{g}_{\beta-\theta} \not\subset \mathrm{Lie}P_J$, $t^{-r-2}\mathfrak{g}_{\theta-\beta}$ indeed belongs to $\mathrm{Lie}P_J^u$. The lemma follows. □

This lemma may also be reformulated as follows: the induced action of $\mathrm{Aut}^+(D)$ on $G_J = P_J/P_J^u$ is trivial.

Let

$$X_{A,J} = \{g \in G(F) | \mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r}d\log(g^{-1}) \in \mathrm{Lie}P_J\}/P_J \subset G(F)/P_J$$

be the deformed Springer fiber in $G(F)/P_J$. By Lemma 11, this is well-defined. For example, if $J = I_{\mathrm{f}}$, then $X_{A,J} = Y_A$. The natural projection $\pi_J : \mathcal{F}\ell \to G(F)/P_J$ restricts to a map $\pi_J : X_A \to X_{A,J}$.

Let $\widetilde{\mathfrak{g}}_J \xrightarrow{p_J} \mathfrak{g}_J$ be the Grothendieck alteration of $\mathfrak{g}_J$, which classifies pairs consisting of a Borel subalgebra of $\mathfrak{g}_J$ and an element contained in this subalgebra.

Proposition 12. *There is a natural Cartesian diagram*

$$\begin{array}{ccc} X_A & \longrightarrow & [\widetilde{\mathfrak{g}}_J/G_J] \\ {\scriptstyle \pi_J}\downarrow & & \downarrow{\scriptstyle p_J} \\ X_{A,J} & \xrightarrow{\varphi_J} & [\mathfrak{g}_J/G_J] \end{array}$$

Proof. We first construct the morphisms $X_{A,J} \to \mathfrak{g}_J/G_J$. Let $\widetilde{X}_{A,J}$ be the preimage of $X_{A,J}$ under the projection $G(F)/P_J^u \to G(F)/P_J$. We have

$$\widetilde{X}_{A,J} = \{g \in G(F) | \mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g^{-1})g \in \mathrm{Lie}P_J\}/P_J^u$$

By Lemma 11, the map

$$gP_J^u \mapsto \mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g^{-1})g \quad \mod \mathrm{Lie}P_J^u$$

is a well define G_J-equivariant map $\widetilde{X}_{A,J} \to \mathfrak{g}_J$. This gives the desired map $\varphi_J : X_{A,J} \to \mathfrak{g}_J/G_J$.

Let $\widetilde{X}_A := X_A \times_{X_{A,J}} \widetilde{X}_{A,J}$, so that $\widetilde{X}_A$ classifies the pairs $(gI, g'P_J^u), g, g' \in G(F)$ such that $gP_J = g'P_J$ and

$$\mathrm{Ad}_{g^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g^{-1})g \in \mathrm{Lie}I, \quad \mathrm{Ad}_{g'^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g'^{-1})g' \in \mathrm{Lie}P_J.$$

On the other hand, $\widetilde{\widetilde{X}}_A := \widetilde{X}_{A,J} \times_{\mathfrak{g}_J} \widetilde{\mathfrak{g}}_J$ classifies pairs $(gI, g'P_J^u), g \in P_J, g' \in G(F)$ such that

$$\mathrm{Ad}_{g'^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g'^{-1})g' \in \mathrm{Ad}_g(\mathrm{Lie}I) \subset \mathrm{Lie}P_J.$$

Let $(gI, g'P_J^u) \in \widetilde{\widetilde{X}}_A$. We find that

$$\mathrm{Ad}_{(g'g)^{-1}}(\widetilde{A}) - t^{-r}\partial_t((g'g)^{-1})(g'g) = \mathrm{Ad}_{g^{-1}}(\mathrm{Ad}_{g'^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g'^{-1})g') - t^{-r}\partial_t(g^{-1})g$$

is in $\mathrm{Lie}I$. This is because $\mathrm{Ad}_{g'^{-1}}(\widetilde{A}) - t^{-r}\partial_t(g'^{-1})g' \in \mathrm{Ad}_g(\mathrm{Lie}I)$ and $t^{-r}\partial_t(g^{-1})g \in \mathrm{Lie}P_J^u \subset \mathrm{Lie}I$ by Lemma 11. Therefore, $(g'gI, g'P_J^u) \in \widetilde{X}_A$. Conversely, if $(gI, g'P_J^u) \in \widetilde{X}_A$, then $(g'^{-1}gI, g'P_J^u) \in \widetilde{\widetilde{X}}_A$. Therefore, there is a G_J-equivariant isomorphism $\widetilde{X}_A \to \widetilde{\widetilde{X}}_A$ sending $(gI, g'P_J^u) \to (g'^{-1}gI, g'P_J^u)$.

Thus, we obtain a Cartesian diagram

$$\begin{tikzcd} X_A \times_{X_{A,J}} \widetilde{X}_{A,J} \arrow[r] \arrow[d] & \widetilde{\mathfrak{g}}_J \arrow[d] \\ \widetilde{X}_{A,J} \arrow[r] & \mathfrak{g}_J \end{tikzcd}$$

where all morphisms are G_J-equivariant. The proposition follows by taking the G_J-quotients. □

Let $\underline{\mathbb{C}}$ be the constant sheaf on X_A. Then $(\pi_J)_*\underline{\mathbb{C}} = \varphi_J^*(p_J)_*\underline{\mathbb{C}}$. By the Springer theory for finite Weyl group, we obtain an action of W_J (the finite Weyl group of G_J) on $(\pi_J)_*\underline{\mathbb{C}}$. Therefore, we obtain a representation of W_J on $H_*(X_A)$. Following the argument of [Lu] §5.5, we obtain that these representations for all $J \subsetneq I_{\mathrm{aff}}$ give rise to a representation of W_{aff} on $H_*(X_A)$.

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